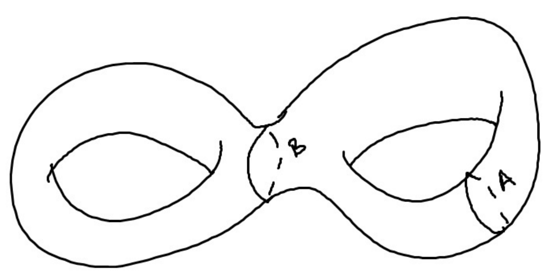
TOPOLOGY III MID-TERM EXAM

RAMESH SREEKANTAN

Each question has the indicated marks adding up to a total of 70 marks. 1 mark will be deducted for not spelling your name and my name correctly.

1. Compute the homology groups $H_n(X, A)$ and $H_n(X, B)$ where X is a closed orientable surface of genus 2 and A and B are the circles shown.(10)



2. For an invertible linear transformation $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ show that the induced map on $H_n(\mathbb{R}^n, \mathbb{R}^n - 0) \simeq \mathbb{Z}$ is $\mathbf{1}$ or $-\mathbf{1}$ according to whether $\det(F)$ is positive or negative. Hint: Use Gaussian elimination to show that the matrix of F can be joined by a path of invertible matrices to a matrix whose diagonal entries are ± 1. (10)

3. Show that

$$H_i(X \times S^n) \simeq H_i(X) \oplus H_{i-n}(X)$$

for all i and all n (where $H_i = 0$ for $i < 0$. by definition). Do this by showing $H_i(X \times S^n) = H_i(X) \oplus H_i(X \times S^n, X \times \{x_0\})$ and that $H_i(X \times S^n, X \times \{x_0\}) \simeq H_{i-1}(X \times S^n, X \times \{x_0\})$.(10)

4. Show that the functors

$$h^n(X) = Hom(H_n(X), \mathbb{Z})$$

do not define a cohomology theory on the category of CW-complexes. (10)

5. If $f : S^n \to S^n$ has degree d, show that

$$f^* : H^n(S^n; G) \to H^n(S^n; G)$$

is given by multiplication by d. (10)

6a. Show there is no map $\mathbb{R}P^n \longrightarrow \mathbb{R}P^m$ inducing a non-trivial map on $H^1(\mathbb{R}P^m; \mathbb{Z}/2\mathbb{Z}) \to H^1(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z})$ if $n > m$. Hint: Use the cup product structure. (10)

6b. Use that to prove the Borsuk-Ulam theorem – show that if there exists $f : S^n \to \mathbb{R}^n$ satisfying $f(x) \neq f(-x)$ for all x. Then if $g : S^n \to S^n$ is defined by

$$g(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|}$$

it violates part (a). (10)